Graduate Comprehensive Examination

Department of Mathematical Sciences

MA541, Probability and Mathematical Statistics II

Answer ALL questions in THREE hours.

May 9, 2017

1. Let $X_1, \ldots, X_n \overset{ind}{\sim} \text{Poisson}(\lambda)$. Consider the class of estimators,

$$T_\omega = \omega \bar{X} + (1-\omega)S^2, 0 \leq \omega \leq 1,$$

where $\bar{X}$ is the sample mean and S^2 the sample variance.

a. What is the size of this class of estimators and what key property does it have?

b. Find the minimum variance of T_ω as a function of ω.

c. Show that $E(S^4) > \lambda/n + \lambda^2$.

2. Let $X_1, \ldots, X_n \mid \mu, \sigma^2 \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$. Consider the following interval estimator $(\bar{X} - kS, \bar{X} + kS)$ of μ, where k is a positive constant, $\bar{X}$ is the sample mean and S^2 is the sample variance.

a. Find the distribution of S.

b. Find the probability content of the interval conditional on $S = s$.

c. Find the probability content of this interval, without conditioning on S, in its simplest form.

3. Let $X_1, \ldots, X_{2n} \mid \theta \overset{ind}{\sim} \text{Uniform}(0, \theta), \theta > 0$. Suppose the first n values are observed and the next n values are missing. Let T denote the largest observation among the missing values.

a. Find a pivotal quantity that is an ancillary statistic.

b. Find the distribution of the pivotal quantity.

c. Find the $100(1-\alpha)\%$ shortest prediction interval for T.

4. An item on a questionnaire asks n respondents to report the value of one of two positive random variables, X or SX. Each respondent actually reports $Z = S^Y X$, where $Y \sim \text{Bernoulli}(p)$, p known. Here, X and S are independent and the distribution of S is completely known, $E(S) = 1$ and $\text{Var}(S) = \gamma^2$. Inference is required about the mean, μ, of X with $\text{Var}(X) = \sigma^2$ known. Let $\hat{\mu}$ denote an unbiased estimator of μ.

a. Obtain a form for $\hat{\mu}$.

b. Find $\text{Var}(\hat{\mu})$.

5. Suppose Y is a random variable from a Weibull distribution with shape parameters λ and scale parameter θ, i.e., $Y \sim \text{WB}(\theta, \lambda)$, with its pdf as

$$f(y \mid \theta, \lambda) = \frac{\lambda y^{\lambda-1}}{\theta^\lambda} \exp(-(\tfrac{y}{\theta})^\lambda), \quad y > 0,$$

where $\theta > 0$ and $\lambda > 0$.

a. Show that if $\lambda = \lambda_0 > 0$ is known, then $\text{WB}(\theta, \lambda_0)$, $\theta > 0$ is a member of the exponential family. What if λ is unknown? Justify.

b. Derive the MLE of the parameter θ when $\lambda = \lambda_0$ is known. For this case, obtain the asymptotic distribution of the MLE of θ.

c. Derive the MLE of the parameters θ and λ when both are unknown.

6. Consider the following pdf

$$f(x, y \mid \sigma) = \frac{\sigma^2}{\pi\sqrt{3}} \exp\left\{-\frac{2\sigma^2}{3}\left[(x-1)^2 + (y-2)^2 - (x-1)(y-2)\right]\right\} \quad \text{for } (x, y) \in R^2,$$

where σ is the unknown parameter. Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are i.i.d. with common pdf $f(x, y \mid \sigma)$.

a. Find the form of the UMP level α test for

$$H_0 : \sigma \le \sigma_0 \text{ versus } H_1 : \sigma > \sigma_0.$$

b. Show that there is no UMP level α test for

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \ne \sigma_0$$

with $0 < \alpha < 1$ fixed.

c. Derive the level α likelihood ratio test for

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \ne \sigma_0$$

in its *simplest implementable form*, with $0 < \alpha < 1$ fixed.